Consensus diffusion in a population

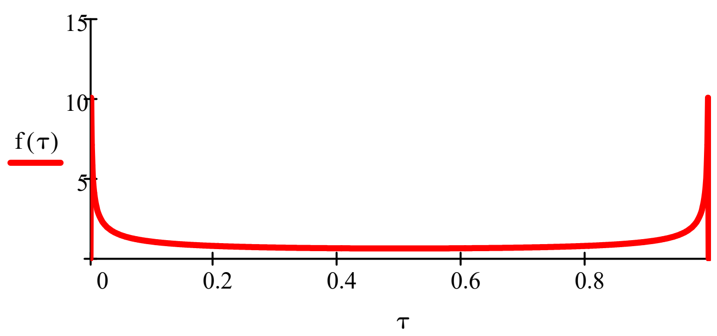
The initial inspiration comes from: **Mark Granovetter** (1978); 'Threshold models of Collective Behavior', *The American Journal of Sociology*, **83**, 6.

Consider a population of voters. Assume that each individual gives his/her consensus to the government according to the fact that at least some people has given.
At time t the percentage of the population that has given consensus is x_t. Each individual has a threshold τ such that if at least the percentage τ is in favour, also that individual decides to be in favor.
A function $f(\tau)$ describes the distribution of these threshold in the population in the sense that the percentage of people with threshold between a and b is given by $\int_a^b f(\tau)\,d\tau$. Of course $\int_0^1 f(\tau)\,d\tau$ is equal to 100% = 1.
Such a function is called *frequency density function* (*fdf* for short).
The percentage of people with threshold between 0 and x is $F(x)$ and it can be expressed as $\int_0^x f(\tau)\,d\tau$.and it bis usually called *cumulative distribution function* (*cdf* for short).

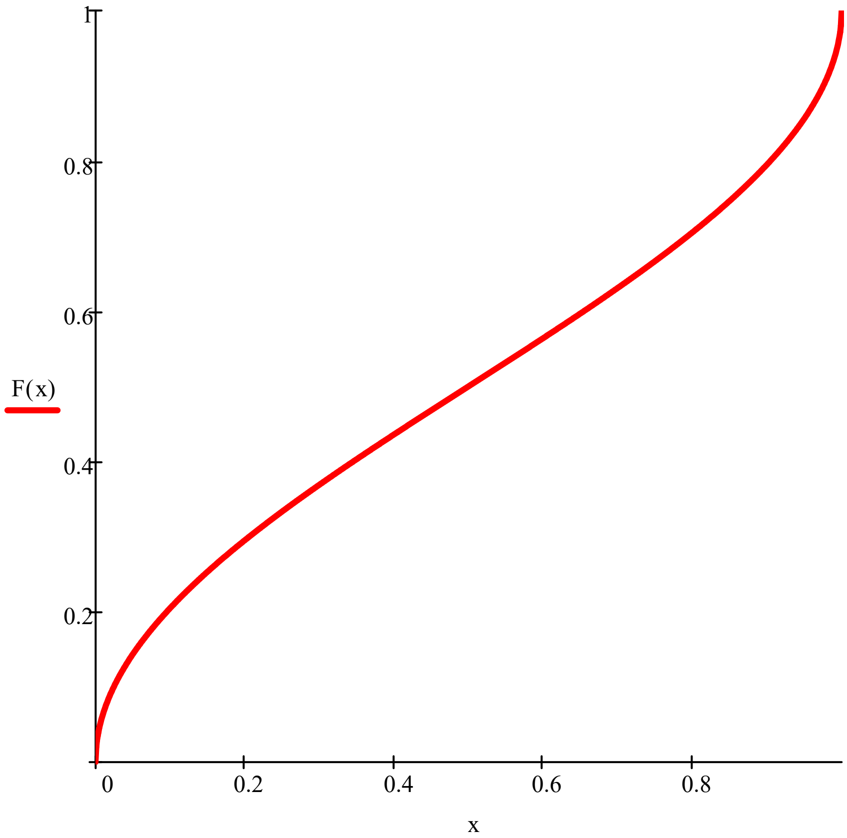
An example could be provided by:

$a := .5 \qquad b := .5$

$f(\tau) := \mathrm{dbeta}(\tau, a, b)$



$$F(x) := \int_0^x f(\tau)\, d\tau$$

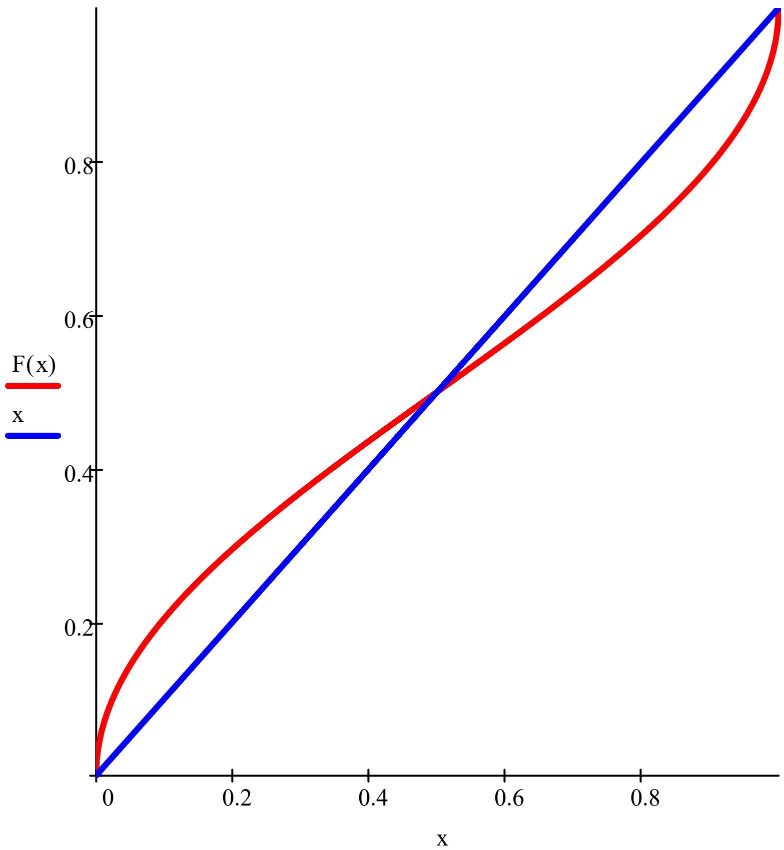


The shape of the cdf is the one indicated in the diagram above under very general conditions.
In the case all the thresholds have the same frequency density, it would be 1.
What we can realistically expect is that low thresholds (very enthusiastic individuals) and high thresholds (very diffident individuals) are not so frequent as average thresholds.
This fact occurs if the frequency density function is unimodal, i.e.; if the fdf first increases and then decreases, which corresponds to the assumption that $f'(\tau) > 0$ for $\tau < \tau^*$ and $f'(\tau) < 0$ for $\tau > \tau^*$. The inflection point of the cdf is preciseal y at τ^*. In our example $\tau\_star = 0.5$

We can imagine a very simple motion law for consensus using this framework.
The consensus percentage at time $t - 1$ determines the percentage of consensus at time t through the cdf: if today the consensus is at the level x_today, the consensus level tomorrow will be x_tomorrow = F(x_today).

We would like to investigate the behavior of consensus as a dynamic system, over a certain time horizon.

Before looking at the numerical results (which depend on the specification of the fdf), we can look at the phase diagram, which turns out to be very illuminating.



The system has three equilibria. Two out of them are trivial. According to the model, if the initial consensus is 0 or 100%, the system does not evolve.
The third equilibrium corresponds to the intersection between the phase curve and the bisector. In our example it is at:

$EQ = 0.5$

The two trivial equilibria are locally stable, while EQ is unstable.

If the system starts from an initial consensus below EQ it will go to 0,
if it starts from an initial position above it will go to 100%.

Numerical example

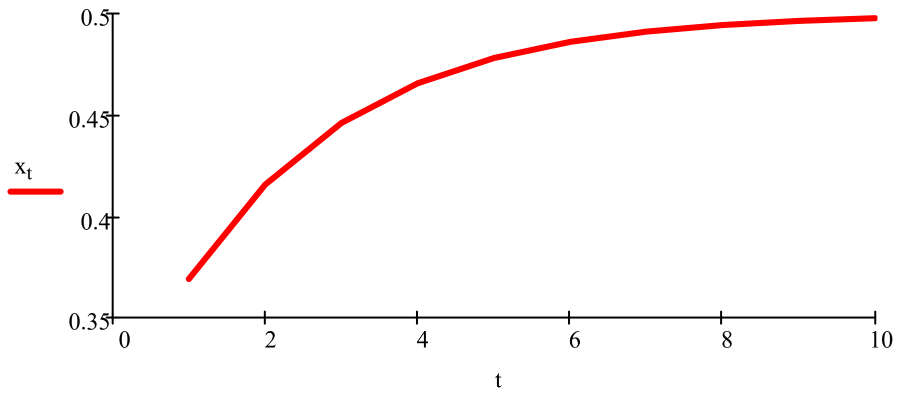
$T := 10$

$t := 1 .. T$

We choose a consensus level at time 0 $\quad x_0 := 30\cdot\%$

the motion law is: $x_t := F(x_{t-1})$

The consequent behavior is:



$$\tau\_star := \frac{a}{a + b}$$

$x := .3 \qquad EQ := \mathrm{root}(x - F(x), x)$