Predator - Prey models

This model is politically relevant when designing environmental policies.
It was independently proposed in 1925 by Alfred J. Lotka, an American born in Ucraine, and in 1926 by Vito Volterra, one among the best Italian mathematicians, to study the interaction between different species.
In the Volterra case, his interest for the problem was raised by his son-in-law, a biologist, who asked him to provide an interpretation about the cyclical presence of some types of fishes in the Adriatic Sea.
Volterra's answer clarified the dynamic reasons of the phenomenon and providend biology with new tools.
The model has been used not only in biology, but also in economics and in marketing.
The original version was a continuous-time one.
Here you find the discrete-time counterpart.
The version you find is better than the original one, because an unnnecessary assumption has been removed.

Your teacher is far from being young and his cultural reference system is certainly obsolete.

For him, when he was young, an important writer was the Nobel Prize for Literature: John Steinbeck.
His short and (amazing) novel "Of Mice and Men" suggests a subtitle for this application:

Of Mice and Cats

Think of a very simple world, where mice and cats are living.
The only food for cats are mice, while mice have no problem in finding food (comments about new food are trivial).
The food attitude of cats links strictly the evolution of the two species.

The story starts from mice: they have a natural growth rate μ which woulld bring them to an exponential growth in a sort of Paradise, where cats are absent. Unfortunately for them cats are present and reduce the mice growth proportionally to the number of mice (to be eaten) and of cats 8eating mice).

The evolution equation for the size of the mice population [m(t) is the number of mice at t] is:

$$m(t+1) = m(t) \,.\, (1+\mu) - \beta \,.\, m(t) \,.\, c(t)$$

The analogous equation for cats captures the fact that cats have a precise system of values: first food, second sex, therefore the growth rate of the size of their population turns out to depend on the number of available mice.
The standard Lotka-Volterra model is grounded on the assumption that there is some survival threshold s for the number of available mice, separating positive and negative growth rates.
This is the version of the model available on WIKIPEDIA, but it is a bit weak.
A more reasonable assumption is that such threshold s depends on the number of cats: many cats need many mice, few cats need few mice.

The evolution equation for the size c(t) of the cat population could be:

$c(t+1) = c(t) . [1 + \gamma . (m(t) - \sigma\, c(t))]$

Note that in the case of mouse scarcity the growth rate of cats is negative (no food => r sex => no puppies...].

Two aspects of this model turn out to be interesting:
(1) - Compute the evolution of the system starting from some given position;
(2) - Find possible equilibria and investigate their stability (not easy).

We attack the first problem.

Time horizon: $T := 50$

Clock: $t := 1 .. T$

Mice period growth rate: $\mu := 100 \cdot \%$

Efficiency of cats in capturing mice: $\beta := .001$

Period growth rate of cats, to be corrected taking into account the number of available mice and of cats $\gamma := .3$

necessary number of mice for the survival a standard cat $\sigma := .1$

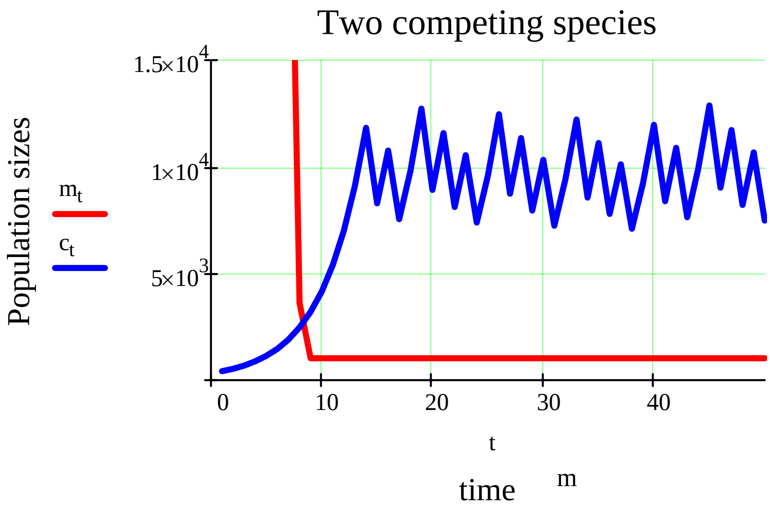
We start from an initial population of both the species

Mice: $m_0 := 10000$

Cats: $c_0 := 300$

The motion law for the system is:

$$\begin{pmatrix} m_t \\ c_t \end{pmatrix} := \begin{bmatrix} \mathrm{ppos}\left[\left[m_{t-1}\cdot(1+\mu) - \beta\cdot\left(m_{t-1} > 300\right)\cdot m_{t-1}\cdot c_{t-1}\right]\right] \\ c_{t-1}\cdot\left[1+\gamma\cdot p\left(\left(m_{t-1} - \sigma\cdot c_{t-1}\right)\right)\right] \end{bmatrix}$$



$m_{30} = 1 \times 10^3$

As far as equilibria are concerned, we start from the equilibrium equations in M,C:

M = M.(1 + μ) - βMC

C = C (1 + γ.(M - σC))

There are two equilibria

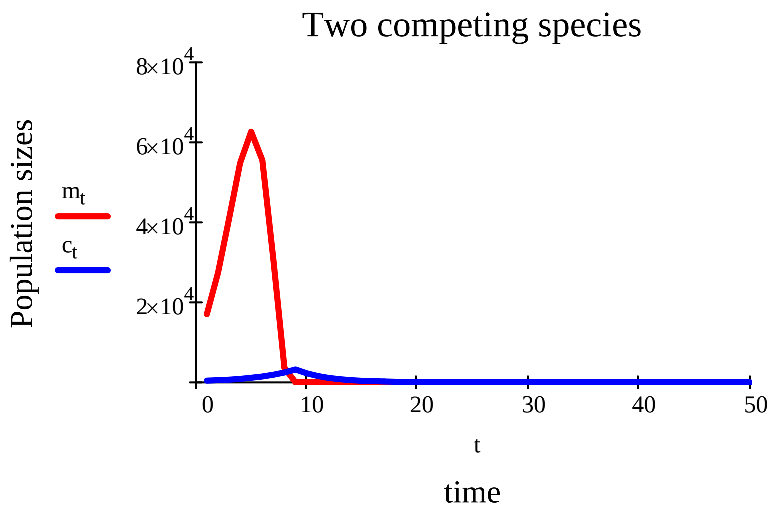
The first equilibrium: $M_1 := 0$ $M_2 := 0$ corresponds to the extinction of the two species

This equilibrium excluded in our numerical model as a condition has been imposed on the minimum number of mice.

The dynamic system without such a restriction would be:

$$\begin{pmatrix} m_t \\ c_t \end{pmatrix} := \begin{bmatrix} \text{pppos}\left[\left[m_{t-1}\cdot(1+\mu) - \beta\cdot\left(m_{t-1} > 300\right)\cdot m_{t-1}\cdot c_{t-1}\right]\right] \\ c_{t-1}\cdot\left[1 + \gamma\cdot p\left(\left(m_{t-1} - \sigma\cdot c_{t-1}\right)\right)\right] \end{bmatrix}$$

and sad dynamics:



The other equilibrium point is: $M2 := \sigma\cdot\frac{\mu}{\beta}$ $\quad C2 := \frac{\mu}{\beta}$

$$\begin{pmatrix} M2 \\ C2 \end{pmatrix} = \begin{pmatrix} 100 \\ 1000 \end{pmatrix}$$

It is intuitive as, in this case the steady-state number of new mice:

$$\mu\cdot M2 = 100$$

and the steady state number of eaten mice

$$\beta\cdot M2\cdot C2 = 100$$

do balance.

Here is the analysis:

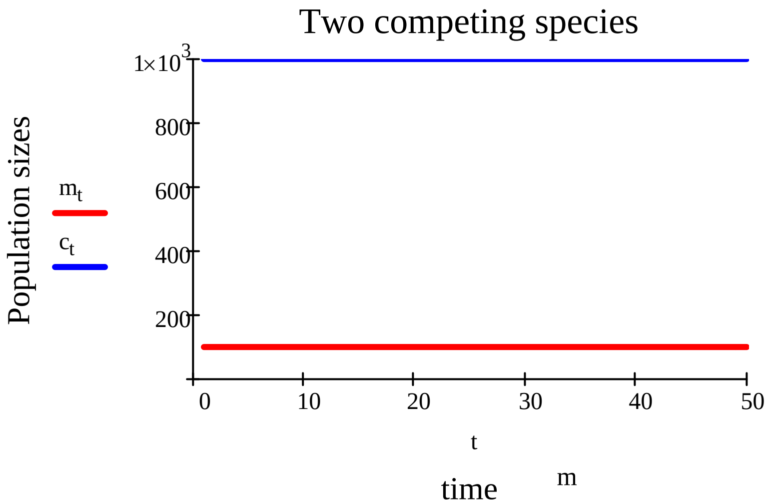
Mice: $m_0 := 100$

Cats: $c_0 := 1000$

The motion law for the system is:

$$\begin{pmatrix} m_t \\ c_t \end{pmatrix} := \begin{bmatrix} \mathrm{ppos}\left[\left[m_{t-1}\cdot(1+\mu) - \beta\cdot m_{t-1}\cdot c_{t-1}\right]\right] \\ c_{t-1}\cdot\left[1+\gamma\cdot p\left(\left(m_{t-1}-\sigma\cdot c_{t-1}\right)\right)\right] \end{bmatrix}$$

The dynamics is... far from being dynamic



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$$ppos(x) := x + (1000 - x)\cdot(x < 0)$$

$$p(x) := x - (x - 1)\cdot(x > 1) + (-1 - x)\cdot(x < -1)$$

$$\text{pppos}(x) := \frac{x + |x|}{2}$$